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2002 J. Phys. A: Math. Gen. 35 3467

(http://iopscience.iop.org/0305-4470/35/15/308)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 3467–3478 PII: S0305-4470(02)29188-X

On the coefficients of differentiated expansions and derivatives of Jacobi polynomials

E H Doha

Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt

E-mail: eiddoha@frcu.eun.eg

Received 26 September 2001, in final form 18 January 2002
Published 5 April 2002
Online at stacks.iop.org/JPhysA/35/3467

Abstract

A formula expressing explicitly the derivatives of Jacobi polynomials of any degree and for any order in terms of the Jacobi polynomials themselves is proved. Another explicit formula, which expresses the Jacobi expansion coefficients of a general-order derivative of an infinitely differentiable function in terms of its original Jacobi coefficients, is also given. The results for the special case of ultraspherical polynomials are considered. The results for Chebyshev polynomials of the first and second kinds and for Legendre polynomials are also noted.

An application of how to use Jacobi polynomials for solving ordinary and partial differential equations is described.

PACS number: 02.30.Gp

Mathematics Subject Classification: 42C10; 33A50; 65L05; 65L10

1. Introduction

Classical orthogonal polynomials are used extensively for the numerical solution of differential equations in spectral and pseudospectral methods (see for instance Ben-Yu 1998, Coutsias *et al* 1996, Doha 1990, 2000, Doha and Helal 1997, Doha and Al-Kholi 2001, Haidvogel and Zang 1979, Siyyam and Syam 1997). In particular, Lewanowicz (1986, 1991, 1992) has presented three different methods for obtaining recurrence relations for the expansion coefficients in Jacobi series solutions of linear ordinary differential equations with polynomial coefficients. Solutions of such recurrence relations enable one to obtain spectral approximations in Jacobi series expansions for the differential equations under consideration.

The importance of Sturm–Liouville problems for spectral methods lies in the fact that the spectral approximation of the solution of a differential equation is usually regarded as a finite expansion of eigenfunctions of a suitable Sturm–Liouville problem.

It is proven that the Jacobi polynomials are precisely the only polynomials arising as eigenfunctions of a singular Sturm–Liouville problem (see Canuto *et al* 1988, section 9.2). This class of polynomials comprises all the polynomial solutions to singular Sturm–Liouville problems on $[-1, 1]$. We have therefore motivated our interest in Jacobi polynomials.

If these polynomials are used as basis functions, then the rate of decay of the expansion coefficients is determined by the smoothness properties of the function being expanded and not by any special boundary conditions satisfied by the function itself. If the function of interest is infinitely differentiable, then the nth expansion coefficient will decrease faster than any finite power of $(1/n)$ as $n \to \infty$ (cf Gottlieb and Orszag 1977).

For the spectral Galerkin or spectral collocation methods; explicit formulae for the expansion coefficients of the derivatives in terms of the original expansion coefficients of the function are often needed. Also explicit expressions for the derivatives of the basis functions in terms of the basis functions themselves are required.

Two explicit formulae expressing the Chebyshev (Legendre) coefficients of a general-order derivative of an infinitely differentiable function in terms of its Chebyshev (Legendre) coefficients are given by Karageorghis (1988a) and by Phillips (1988).

A more general formula for ultraspherical coefficients is given by Karageorghis and Phillips (1989/1992). Such a formula has been stated in a more compact form and proved in a simple way by Doha (1991). A formula expressing explicitly the derivatives of ultraspherical polynomials in terms of ultraspherical polynomials themselves is also given—with its important special cases—for Chebyshev and Legendre polynomials by Doha (1991).

A more general situation which often arises in the numerical solution of differential equations with polynomial coefficients in spectral and pseudospectral methods is the evaluation of the expansion coefficients of the moments of high-order derivatives of infinitely differentiable functions. A formula for the shifted Chebyshev coefficients of the moments of general-order derivatives of an infinitely differentiable function is given by Karageorghis (1988b). Corresponding results for Chebyshev polynomials of the first and second kinds, Legendre polynomials and ultraspherical polynomials are given by Doha (1994), Doha and El-Soubhy (1995) and Doha (1998) respectively.

Up to now, and to the best of our knowledge, many formulae corresponding to those mentioned previously are not known and are traceless in the literature for the Jacobi expansions. This also motivates our interest in such polynomials. Another motivation is that the theoretical and numerical analysis of numerous physical and mathematical problems very often requires the expansion of an arbitrary polynomial or the expansion of an arbitrary function with its derivatives and moments into a set of orthogonal polynomials. This is in particular true for Jacobi polynomials.

In section 2 we give relevant properties of Jacobi polynomials and in section 3 we describe how they are used to solve boundary value problems with the Galerkin method. In section 4 we prove the main results of the paper, which are the following.

(i) An explicit expression for the derivatives of Jacobi polynomials of any degree and for any order in terms of the Jacobi polynomials themselves.

(ii) An explicit formula for the coefficients of a general-order derivative of an expansion in Jacobi polynomials in terms of the coefficients of the original expansion.

In section 5 we explain how the Jacobi polynomials are used to solve differential equations by the collocation method. Two numerical examples for the solution of linear partial differential equations in two independent variables by using the collocation method are given in section 6.

2. Some relevant properties of Jacobi polynomials

The Jacobi polynomials associated with the real parameters ($\alpha > -1, \beta > -1$) (see Szegö 1985) are a sequence of polynomials $P_n^{(\alpha,\beta)}(x) (n = 0, 1, 2, \ldots)$, each respectively of degree n, satisfying the orthogonality relation

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x)\,\mathrm{d}x = \begin{cases} 0 & m \neq n, \\ h_n & m = n, \end{cases}$$

where

$$h_n = \frac{2^{\alpha+\beta+1}\,\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\,n!\,\Gamma(n+\alpha+\beta+1)}. \tag{1}$$

The Jacobi polynomials are eigenfunctions of the following singular Sturm–Liouville problem:

$$(1-x^2)u''(x) + [\beta - \alpha - (\alpha+\beta+2)x]u'(x) + n(n+\alpha+\beta+1)u(x) = 0.$$

A consequence of this is that spectral accuracy can be achieved for expansions in Jacobi polynomials. For our present purposes it is convenient to standardize the Jacobi polynomials so that

$$P_n^{(\alpha,\beta)}(1) = \frac{\Gamma(n+\alpha+1)}{n!\,\Gamma(\alpha+1)}, \qquad P_n^{(\alpha,\beta)}(-1) = \frac{(-1)^n\,\Gamma(n+\beta+1)}{n!\,\Gamma(\beta+1)}.$$

In this form the polynomials may be generated using the recurrence relation

$$\begin{aligned} 2(n+1)(n+\lambda)(2n+\lambda-1)P_{n+1}^{(\alpha,\beta)}(x) &= (2n+\lambda-1)_3\, x P_n^{(\alpha,\beta)}(x) \\ &\quad + (\alpha^2-\beta^2)(2n+\lambda)P_n^{(\alpha,\beta)}(x) - 2(n+\alpha)(n+\beta)(2n+\lambda+1)P_{n-1}^{(\alpha,\beta)}(x), \\ &\quad (n = 1, 2, \ldots), \end{aligned} \tag{2}$$

starting from $P_0^{(\alpha,\beta)}(x) = 1$ and $P_1^{(\alpha,\beta)}(x) = \frac{1}{2}[\alpha - \beta + (\lambda+1)x]$, or obtained from Rodrigue's formula

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n\,n!}(1-x)^{-\alpha}(1+x)^{-\beta}D^n[(1-x)^{\alpha+n}(1+x)^{\beta+n}], \tag{3}$$

where

$$\lambda = \alpha+\beta+1, \qquad (a)_k = \Gamma(a+k)/\Gamma(a), \qquad D \equiv \frac{\mathrm{d}}{\mathrm{d}x}.$$

Of these polynomials, the most commonly used are the Gegenbauer ultraspherical polynomials $C_n^{(\alpha)}(x)$, the Chebyshev polynomials $T_n(x)$ of the first kind, the Legendre polynomials $P_n(x)$ and the Chebyshev polynomials of the second kind $U_n(x)$. These orthogonal polynomials are interrelated to the Jacobi polynomials by the following relations:

$$\begin{aligned} C_n^{(\alpha)}(x) &= \frac{n!\,\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(n+\alpha+\frac{1}{2})} P_n^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(x), \qquad & T_n(x) &= \frac{n!\,\sqrt{\pi}}{\Gamma\left(n+\frac{1}{2}\right)} P_n^{(-\frac{1}{2},-\frac{1}{2})}(x), \\ P_n(x) &= P_n^{(0,0)}(x), & U_n(x) &= \frac{(n+1)!\,\sqrt{\pi}}{2\,\Gamma\left(n+\frac{3}{2}\right)} P_n^{(\frac{1}{2},\frac{1}{2})}(x). \end{aligned} \tag{4}$$

Suppose now we are given a function $f(x)$ which is infinitely differentiable in the closed interval $[-1, 1]$; then we may represent it in the form

$$f(x) = \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(x). \tag{5}$$

Further, let $a_n^{(q)}$ denote the Jacobi coefficients of the qth derivative of $f(x)$; that is

$$f^{(q)}(x) = \frac{\mathrm{d}^q f(x)}{\mathrm{d}x^q} = \sum_{n=0}^{\infty} a_n^{(q)} P_n^{(\alpha,\beta)}(x). \tag{6}$$

It is possible to derive a recurrence relation involving the Jacobi coefficients of successive derivatives of $f(x)$. Let us write

$$\frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=0}^{\infty} a_n^{(q-1)} P_n^{(\alpha,\beta)}(x) = \sum_{n=0}^{\infty} a_n^{(q)} P_n^{(\alpha,\beta)}(x),$$

then use of the identity

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) = \frac{2}{(n+\lambda-1)(2n+\lambda-1)_3}&[(n+\lambda-1)_2\,(2n+\lambda-1)DP_{n+1}^{(\alpha,\beta)}(x) \\ &+ (\alpha-\beta)(n+\lambda-1)(2n+\lambda)DP_n^{(\alpha,\beta)}(x) \\ &- (n+\alpha)(n+\beta)(2n+\lambda+1)DP_{n-1}^{(\alpha,\beta)}(x)], \qquad n \geqslant 1, \end{aligned} \tag{7}$$

leads to the recurrence relation

$$\begin{aligned} &\frac{(n+\lambda-1)}{(2n+\lambda-1)(2n+\lambda-2)}\,a_{n-1}^{(q)} + \frac{(\alpha-\beta)}{(2n+\lambda+1)(2n+\lambda-1)}\,a_n^{(q)} \\ &\quad - \frac{(n+\alpha+1)(n+\beta+1)}{(2n+\lambda+2)(2n+\lambda+1)(n+\lambda)}\,a_{n+1}^{(q)} = \frac{1}{2}\,a_n^{(q-1)}, \qquad q \geqslant 1, n \geqslant 1, \end{aligned} \tag{8}$$

where $a_n^{(0)} = a_n$.

3. The Galerkin method for a boundary value problem

Consider the solution of the differential equation

$$u''(x) + \gamma\, u(x) = g(x), \qquad x \in [-1, 1], \tag{9}$$

subject to $u(\pm 1) = 0$, where γ is a known scalar. Suppose that we approximate $u(x)$ by a truncated series expansion of Jacobi polynomials

$$u_N(x) = \sum_{n=2}^{N} a_n \left[P_n^{(\alpha,\beta)}(x) - \left(\frac{1+x}{2}\right) P_n^{(\alpha,\beta)}(1) - \left(\frac{1-x}{2}\right) P_n^{(\alpha,\beta)}(-1) \right]; \tag{10}$$

we seek to determine a_n using the Galerkin method. Note that the boundary conditions are automatically satisfied. It is not difficult to put $u_N(x)$ in the form

$$u_N(x) = \sum_{n=2}^{N} a_n [P_n^{(\alpha,\beta)}(x) - r_n\; P_0^{(\alpha,\beta)}(x) - s_n\; P_1^{(\alpha,\beta)}(x)], \tag{11}$$

where

$$\begin{aligned} r_n &= [(\beta+1)(\alpha+1)_n + (-1)^n(\alpha+1)(\beta+1)_n]/n!\,(\lambda+1), \\ s_n &= [(\alpha+1)_n - (-1)^n(\beta+1)_n]/n!\,(\lambda+1). \end{aligned}$$

Since $u_N''(x)$ is a polynomial of degree at most $N-2$ we may write

$$u_N''(x) = \sum_{n=0}^{N-2} a_n^{(2)} P_n^{(\alpha,\beta)}(x). \tag{12}$$

The coefficients a_n are chosen so that $u_N(x)$ satisfies

$$u_N''(x) + \gamma\, u_N(x) = g_N(x), \tag{13}$$

where

$$g_N(x) = \sum_{n=0}^{N} d_n P_n^{(\alpha,\beta)}(x).$$

Substituting (11) and (12) into (13), multiplying by $(1 - x)^\alpha (1 + x)^\beta P_m^{(\alpha,\beta)}(x), m = 0, 1, \ldots, N - 2$, and integrating over the interval $[-1, 1]$ yields

$$\begin{aligned} &a_0^{(2)} - \gamma \sum_{n=2}^{N} r_n a_n = d_0, \\ &a_1^{(2)} - \gamma \sum_{n=2}^{N} s_n a_n = d_1, \\ &a_m^{(2)} + \gamma a_m = d_m, \qquad m = 2, \ldots, N-2. \end{aligned} \tag{14}$$

Thus there are $(N - 1)$ equations for the $(N - 1)$ unknowns $a_2, a_3, \ldots, a_N$. In order to obtain a solution to (13), it is only necessary to solve (14) for the $(N - 1)$ unknown coefficients a_n $(2 \leqslant n \leqslant N)$.

The coefficients $a_n^{(2)}$ of the second derivative of the approximation $u_N(x)$ are related to the coefficients a_n of $u_N(x)$ by invoking (8) with $q = 1$ and 2. In the next section we show how the coefficients of any derivative may be expressed in terms of the original expansion coefficients. This allows us to replace $a_m^{(2)}$ in (14) by an explicit expression in terms of a_n. In this way we can set up a linear system for a_n $(2 \leqslant n \leqslant N)$ which may be solved using standard direct solvers.

Returning to the difference equation (8), and for computing purposes, we see that this equation is not easy to use, since the coefficients on the left-hand side are functions of n. No obvious direct method is available for solving this equation, therefore we resort to the following alternative method that enables us to express $a_n^{(q)}$ in terms of the original expansion coefficients $a_k, k = 0, 1, 2, \ldots$.

4. The derivatives of $P_n^{(\alpha,\beta)}(x)$ and the relation between the coefficients $a_n^{(q)}$ and a_n

The main objective of this section is to prove the following theorem for the derivatives of $P_n^{(\alpha,\beta)}(x)$ and the coefficients $a_n^{(q)}$.

Theorem.

$$D^q P_n^{(\alpha,\beta)}(x) = 2^{-q}(n + \alpha + \beta + 1)_q \sum_{i=0}^{n-q} C_{n-q,i}(\alpha + q, \beta + q, \alpha, \beta) P_i^{(\alpha,\beta)}(x), \tag{15}$$

where

$$C_{n-q,i}(\alpha + q, \beta + q, \alpha, \beta) = \frac{(n + q + \alpha + \beta + 1)_i \; (i + q + \alpha + 1)_{n-i-q} \; \Gamma(i + \alpha + \beta + 1)}{(n - i - q)! \; \Gamma(2i + \alpha + \beta + 1)} \times {}_3F_2\left(\begin{matrix} -n + q + i, & n + i + q + \alpha + \beta + 1, & i + \alpha + 1 \\ i + q + \alpha + 1, & 2i + \alpha + \beta + 2 & \end{matrix} ; 1 \right),$$

and

$$a_n^{(q)} = 2^{-q} \sum_{i=0}^{\infty} (n + i + q + \alpha + \beta)_q \; C_{n+i,n}(\alpha + q, \beta + q, \alpha, \beta) \; a_{n+i+q}, \qquad n \geqslant 0, q \geqslant 1, \tag{16}$$

where

$$C_{n+i,n}(\alpha+q,\beta+q,\alpha,\beta)=\frac{(n+i+2q+\alpha+\beta)_n\ (n+\alpha+q+1)_i\ \Gamma(n+\alpha+\beta+1)}{i!\ \Gamma(2n+\alpha+\beta+1)}\times {}_3F_2\left(\begin{matrix}-i,\ \ 2n+i+2q+\alpha+\beta+1,\ \ n+\alpha+1\\ n+q+\alpha+1,\ \ 2n+\alpha+\beta+2\end{matrix};1\right).$$

Proof. It is well known that $P_n^{(\gamma,\delta)}(x)$ may be defined by

$$P_n^{(\gamma,\delta)}(x)=\frac{(1+\gamma)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n,\ \ n+\gamma+\delta+1;\\ 1+\gamma;\ \ \ \ (1-x)/2\end{matrix}\right)=\frac{(1+\gamma)_n}{(1+\gamma+\delta)_n}\sum_{k=0}^{n}\frac{(1+\gamma+\delta)_{n+k}}{k!\ (n-k)!\ (1+\gamma)_k}\left(\frac{x-1}{2}\right)^k$$

and, accordingly, we obtain by direct differentiation that

$$DP_n^{(\gamma,\delta)}(x)=\frac{(n+\gamma+\delta+1)}{2}\,P_{n-1}^{(\gamma+1,\delta+1)}(x),$$

and therefore

$$D^kP_n^{(\gamma,\delta)}(x)=2^{-k}(n+\gamma+\delta+1)_k\,P_{n-k}^{(\gamma+k,\delta+k)}(x). \tag{17}$$

In order to complete the proof of the theorem we need the following lemma.

Lemma. Suppose

$$P_n^{(\gamma,\delta)}(x)=\sum_{k=0}^{n}C_{nk}(\gamma,\delta,\alpha,\beta)P_k^{(\alpha,\beta)}(x), \tag{18}$$

then

$$C_{nk}(\gamma,\delta,\alpha,\beta)=\frac{(n+\gamma+\delta+1)_k\ (k+\gamma+1)_{n-k}\ \Gamma(k+\alpha+\beta+1)}{(n-k)!\ \Gamma(2k+\alpha+\beta+1)}\times {}_3F_2\left(\begin{matrix}-n+k,\ \ n+k+\gamma+\delta+1,\ k+\alpha+1\\ k+\gamma+1,\ \ 2k+\alpha+\beta+2\end{matrix};1\right). \tag{19}$$

(For proof, see Andrews *et al* 1999.)

From the lemma and the identity (17), we obtain

$$D^iP_n^{(\gamma,\delta)}(x)=2^{-i}(n+\gamma+\delta+1)_i\sum_{k=0}^{n-i}C_{n-i,k}(\gamma+i,\delta+i,\alpha,\beta)P_k^{(\alpha,\beta)}(x),$$

and in particular,

$$D^qP_n^{(\alpha,\beta)}(x)=2^{-q}(n+\alpha+\beta+1)_q\sum_{k=0}^{n-q}C_{n-q,k}(\alpha+q,\beta+q,\alpha,\beta)P_k^{(\alpha,\beta)}(x),$$

which proves the first part of the theorem.

Now, on differentiating (5) q times and making use of (15), we find

$$f^{(q)}(x)=\sum_{n=q}^{\infty}a_nD^qP_n^{(\alpha,\beta)}(x)=2^{-q}\sum_{n=q}^{\infty}(n+\alpha+\beta+1)_q\ a_n\sum_{k=0}^{n-q}C_{n-q,k}(\alpha+q,\beta+q,\alpha,\beta)P_k^{(\alpha,\beta)}(x). \tag{20}$$

Expanding (20) and collecting similar terms, we obtain

$$f^{(q)}(x) = 2^{-q} \sum_{n=0}^{\infty} \left[\sum_{i=0}^{\infty} (n+i+q+\alpha+\beta+1)_q \; C_{n+i,n}(\alpha+q, \beta+q, \alpha, \beta) \; a_{n+i+q} \right] P_n^{(\alpha,\beta)}(x). \tag{21}$$

Identifying (21) with (6) gives immediately

$$a_n^{(q)} = 2^{-q} \sum_{i=0}^{\infty} (n+i+q+\alpha+\beta+1)_q \; C_{n+i,n}(\alpha+q, \beta+q, \alpha, \beta) \; a_{n+i+q},$$

and this completes the proof of the theorem.

Remark 1. It is to be noted here that the formula for $a_n^{(q)}$ given by (16) is the exact solution of the difference equation (8).

Remark 2. In general, the ${}_3F_2$ in the theorem cannot be summed, but it can be summed by Watson's identity (1925) if $\alpha = \beta$.

The particular expressions for the ultraspherical polynomials may be derived as a special case of the theorem. We give this as a corollary to the main theorem.

Corollary (Doha 1991). If $\alpha = \beta$, and each is replaced by $(\alpha - \frac{1}{2})$, then equations (4)–(6) give

$$f(x) = \sum_{n=0}^{\infty} A_n C_n^{(\alpha)}(x), \qquad A_n = \frac{\Gamma\big(n+\alpha+\frac{1}{2}\big)}{n!\,\Gamma\big(\alpha+\frac{1}{2}\big)}\, a_n, \tag{22}$$

$$f^{(q)}(x) = \sum_{n=0}^{\infty} A_n^{(q)} C_n^{(\alpha)}(x), \qquad A_n^{(q)} = \frac{\Gamma\big(n+\alpha+\frac{1}{2}\big)}{n!\,\Gamma\big(\alpha+\frac{1}{2}\big)}\, a_n^{(q)}, \qquad n \geqslant 0, q \geqslant 1, \tag{23}$$

where

$$A_n^{(q)} = \frac{2^q (n+\alpha)\Gamma(n+2\alpha)}{(q-1)!\,n!} \sum_{i=0}^{\infty} \frac{(i+q-1)!\,\Gamma(n+i+q+\alpha)(n+2i+q)!}{i!\,\Gamma(n+i+\alpha+1)\Gamma(n+2i+q+2\alpha)} A_{n+2i+q}, \tag{24}$$

$$D^q C_n^{(\alpha)} = \frac{2^q\, n!}{(q-1)!\,\Gamma(n+2\alpha)} \times \sum_{\substack{i=0 \\ (n+i-q)\ even}}^{n-q} \frac{(i+\alpha)\Gamma(i+2\alpha)\big((n-i+q-2)/2\big)!\,\Gamma\big((n+i+q+2\alpha)/2\big)}{i!\,\big((n-i-q)/2\big)!\,\Gamma\big((n+i-q+2\alpha+2)/2\big)} \times C_i^{(\alpha)}(x). \tag{25}$$

Proof. In this special case, relation (16) takes the form

$$a_n^{(q)} = 2^{-q} \sum_{i=0}^{\infty} (n+i+q+2\alpha)_q \; C_{n+i,n}\big(q+\alpha-\tfrac{1}{2}, q+\alpha-\tfrac{1}{2}, \alpha-\tfrac{1}{2}, \alpha-\tfrac{1}{2}\big) a_{n+i+q}, \tag{26}$$

where

$$C_{n+i,n}\big(q+\alpha-\tfrac{1}{2}, q+\alpha-\tfrac{1}{2}, \alpha-\tfrac{1}{2}, \alpha-\tfrac{1}{2}\big) = \frac{(n+i+2q+2\alpha)_n \big(n+q+\alpha+\frac{1}{2}\big)_i \Gamma(n+2\alpha)}{i!\,\Gamma(2n+2\alpha)} \times {}_3F_2\left(\begin{matrix} -i,\ 2n+i+2q+2\alpha,\ n+\alpha+\frac{1}{2} \\ n+q+\alpha+\frac{1}{2},\ 2n+2\alpha+1 \end{matrix}; 1\right). \tag{27}$$

Watson (1925) proved that, when i is an even positive integer,

$$ {}_3F_2\left(\begin{matrix} -i,\ i+2\mu+2\nu-1,\ \mu \\ 2\mu, \qquad \mu+\nu \end{matrix};1\right) = \frac{i!\,\Gamma\big(\mu+\frac{i}{2}\big)\,\Gamma\big(\nu+\frac{i}{2}\big)\,\Gamma(2\mu)\,\Gamma(\mu+\nu)}{\big(\frac{i}{2}\big)!\,\Gamma\big(\mu+\nu+\frac{i}{2}\big)\,\Gamma(2\mu+i)\,\Gamma(\mu)\,\Gamma(\nu)}, \tag{28} $$

and when i is an odd integer the sum of the hypergeometric series is zero, and consequently relation (27) takes the form

$$ \begin{aligned} &C_{2i+n,n}\big(q+\alpha-\tfrac{1}{2}, q+\alpha-\tfrac{1}{2}, \alpha-\tfrac{1}{2}, \alpha-\tfrac{1}{2}\big) \\ &\quad = \big((i+q-1)!\,(2n+2\alpha)\Gamma(n+2\alpha)\Gamma\big(n+i+\alpha+\tfrac{1}{2}\big)\Gamma(2n+2i+2q+2\alpha) \\ &\qquad \times \Gamma\big(n+2i+q+\alpha+\tfrac{1}{2}\big)\big)\big/\big\{i!\,(q-1)!\,\Gamma\big(n+\alpha+\tfrac{1}{2}\big)\Gamma\big(n+i+q+\alpha+\tfrac{1}{2}\big) \\ &\qquad \times \Gamma(n+2i+2q+2\alpha)\Gamma(2n+2i+2\alpha+1)\big\}. \end{aligned} \tag{29} $$

Substitution of (29) into (26) with the aid of (23) yields

$$ A_n^{(q)} = \frac{2^q(n+\alpha)\,\Gamma(n+2\alpha)}{n!\,(q-1)!} \sum_{i=0}^{\infty} \frac{(i+q-1)!\,\Gamma(n+i+q+\alpha)\,(n+2i+q)!}{i!\,\Gamma(n+i+\alpha+1)\,\Gamma(n+2i+q+2\alpha)}\,A_{n+2i+q}, \tag{30} $$

which completes the proof of relation (24). The proof of formula (25) is similar to that of (24).

It is worthy of note that formula (30) is in complete agreement with that obtained by Doha (1991, formula (17), p 118). In particular, the special cases may be obtained from (30) directly for the Chebyshev polynomials of the first and second kinds by taking $\alpha = 0, 1$ respectively, and for Legendre polynomials by taking $\alpha = \frac{1}{2}$. These are given explicitly by Doha (1991, formulae (18)–(20), pp 118–9, and formulae (26)–(31), p 120).

5. Use of Jacobi polynomials to solve differential equations

Consider the linear ordinary differential equation of order n of the form

$$ \sum_{i=0}^{n} f_i(x) D^i y(x) = g(x), \tag{31} $$

where $f_i(x)$ and $g(x)$ are functions of x only. Suppose the equation to be solved is in interval $[-1, 1]$ subject to n linear boundary conditions, and assume we approximate $y(x)$ by a truncated expansion of Jacobi polynomials

$$ y(x) = \sum_{j=0}^{N} a_j P_j^{(\alpha,\beta)}(x), \tag{32} $$

where N is the degree of approximation and $a_0, a_1, \ldots, a_N$ are unknown coefficients to be determined. Substituting (32) into (31) yields

$$ \sum_{i=0}^{n} \left\{ f_i(x) \sum_{j=0}^{N} a_j D^i P_j^{(\alpha,\beta)}(x) \right\} = g(x), \tag{33} $$

which may be written in the form

$$ \sum_{j=0}^{N} \left\{ a_j \sum_{i=0}^{n} f_i(x) D^i P_j^{(\alpha,\beta)}(x) \right\} = g(x). \tag{34} $$

The boundary conditions associated with (31) give rise to n equations connecting the coefficients a_j, and the remaining equations may be obtained in two ways.

(i) We may equate the coefficients of the various $P_j^{(\alpha,\beta)}(x)$ after expanding both sides of (34) in Jacobi series.

(ii) We may collocate at $m = N - n$ selected points in $(-1, 1)$.

The system of equations obtained from the collocation is of the form

$$\sum_{j=0}^{N}\left\{a_j \sum_{i=0}^{n} f_i(x_k) D^i P_j^{(\alpha,\beta)}(x_k)\right\} = g(x_k), \qquad k = 1, 2, \ldots, m, \tag{35}$$

where x_k are the collocation points, which are usually chosen at the zeros of $P_m^{(\alpha,\beta)}(x)$ (see Canuto *et al* 1988, appendix C). Since the derivatives $D^i P_j^{(\alpha,\beta)}(x)$ are now expressible explicitly in terms of $P_k^{(\alpha,\beta)}$, the problem of computing them is solved by using formula (15). Therefore, the resulting linear system obtained from (33) and the n linear boundary conditions can easily be solved using standard direct solvers.

The method just described is easily extended to higher dimensions. Consider, for example, the second-order partial differential equation

$$A_1(x, y)u_{xx} + A_2(x, y)u_{xy} + A_3(x, y)u_{yy} + A_4(x, y)u_x + A_5(x, y)u_y + A_6(x, y)u = f(x, y), \tag{36}$$

where the coefficients $A_1, A_2, \ldots, A_6$ and f are functions of x and y only. Suppose the solution of the equation is required in the square $S(-1 \leqslant x, y \leqslant 1)$, subject to general linear boundary conditions of the form

$$B_1(x, y)u_x + B_2(x, y)u_y + B_3(x, y)u = g(x, y), \tag{37}$$

on the sides of the square S.

Suppose the function $u(x, y)$ can be approximated by the double finite Jacobi series

$$u(x, y) = \sum_{m=0}^{M}\sum_{n=0}^{N} a_{mn} P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(y), \tag{38}$$

for sufficiently large values of the integers M and N. Since $u(x, y)$ satisfies (36) we have approximately

$$\begin{aligned}\sum_{m=0}^{M}\Bigg\{\sum_{n=0}^{N} a_{mn}[&A_1 D_x^2 P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(y) + A_2 D_x P_m^{(\alpha,\beta)}(x) D_y P_n^{(\alpha,\beta)}(y) \\ &+ A_3 P_m^{(\alpha,\beta)}(x) D_y^2 P_n^{(\alpha,\beta)}(y) + A_4 D_x P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(y) \\ &+ A_5 P_m^{(\alpha,\beta)}(x) D_y P_n^{(\alpha,\beta)}(y) + A_6 P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(y)]\Bigg\} = f(x, y).\end{aligned} \tag{39}$$

On collocating equation (39) at $(M-1)(N-1)$ distinct points (x_i, y_j), $i = 1, 2, \ldots, M-1$, $j = 1, 2, \ldots, N-1$, in S, there results a set of $(M-1)(N-1)$ linear equations for the coefficients a_{mn}. If we now collocate equation (37) at $2(M+N)$ points on the sides of the square S, we find the remaining equations for the unique determination of the coefficients a_{mn}.

As in ordinary differential equations, the derivatives of Jacobi polynomials occurring in (39) are computed by use of (15), and numbers x_i and y_j are chosen at the zeros of the appropriate Jacobi polynomials.

6. Numerical examples

The collocation method is applied to two test examples in linear partial differential equations in two independent variables. Some of the results obtained are listed below.

Table 1. Non-zero coefficients ($\times 10^{10}$) in the approximate solution of example 1.

n	$m=0$	$m=2$	$m=4$	$m=6$	$m=8$	$m=10$	$m=12$
0	3723 325 062	−10 505 287 448	807 541 643	−20 865 481	269 309	−2083	12
1	−2234 483 264	6 304 549 995	−484 630 872	12 522 022	−161 615	1249	0
2	−296 510 707	836 599 047	−64 309 385	1 661 640	−21 441	166	0
3	30 695 066	−86 605 517	6 657 369	−172 011	2216	−18	0
4	761 026	−2 147 211	165 057	−4 264	56	0	0
5	−85 984	242 607	−18 650	486	0	0	0
6	−208	591	−49	0	0	0	0
7	95	−267	17	0	0	0	0
8	0	0	0	0	0	0	0

Table 2. Non-zero coefficients ($\times 10^{8}$) in the approximate solution of example 2.

n	$m=0$	$m=2$	$m=4$	$m=6$	$m=8$	$m=10$
0	3309 937	8 607 872	−279 852	−20 484	−3 687	−698
1	−296 674	824 683	−44 156	−2 092	−163	−34
2	−7474 468	20 059 929	−95 544	−79 639	−17 015	−3314
3	360 573	−1 018 103	78 163	−326	−358	−73
4	−1664 182	3 909 563	721 525	14 266	−9 963	−2595
5	130 499	−356 233	7 683	4 413	0	−69
6	−196 537	161 854	409 622	94 441	11 761	1258
7	15 319	−32 488	−13 339	1 778	511	54
8	−31 846	−76 016	116 262	92 707	32 568	7109
9	1 801	1 553	−7 808	1 697	467	249
10	−7 804	29 618	17 871	39 374	21 213	5540

Example 1. The current $J(x,t)$ in an insulated cable of resistance R, capacitance $1/R$ and self-inductance $4R$ at a time t satisfies the hyperbolic partial differential equation

$$J_{xx} - 4J_{tt} - J_t = 0.$$

This problem is solved in the rectangle $0 \leqslant x \leqslant 2$, $0 \leqslant t \leqslant 1$, subject to the boundary conditions

$$J(x,0) = -4\,J_t(x,0) = \sin\frac{\pi x}{2}, \qquad 0 \leqslant x \leqslant 2,$$
$$J(0,t) = J(2,t) = 0, \qquad 0 \leqslant t \leqslant 1.$$

A solution of the form

$$J(x,t) = \sum_{n=0}^{8}\sum_{m=0}^{12} a_{mn} P_m^{(\alpha,\beta)}(x-1)\; P_n^{(\alpha,\beta)}(2t-1)$$

is assumed over the rectangle. In particular, and for the case $\alpha = \beta = -\frac{1}{2}$, the coefficients a_{mn} based on collocation at the Chebyshev points $\bigl(1+\cos(i\pi/8), \frac{1}{2}\bigl(1+\cos(j\pi/12)\bigr)\bigr)$, $i = 0(1)8$, $j = 0(1)12$, are shown in table 1; the coefficients not shown are less than 5×10^{-11} in magnitude. Values computed from the solution agreed with the analytical solution given by Collatz (1966, p 326),

$$J(x,t) = \mathrm{e}^{-t/8}\left(\cos\frac{\mu t}{8} - \frac{1}{\mu}\sin\frac{\mu t}{8}\right)\sin\frac{\pi x}{2}, \qquad \mu = \sqrt{4\pi^2 - 1},$$

to nine decimal places.

Example 2. Another problem is the solution of the elliptic equation

$$u_{xx} + u_{yy} + \frac{3}{5-y}\, u_y = -1,$$

in the rectangle $|x| = \frac{1}{2}$, $|y| = 1$, with $u = 0$ on the boundary of the rectangle (see Collatz 1966, pp 357–8). A solution of the form

$$u(x, y) = \sum_{n=0}^{10}\sum_{m=0}^{10} a_{mn} P_m^{(-\frac{1}{2},-\frac{1}{2})}(2x)\; P_n^{(-\frac{1}{2},-\frac{1}{2})}(y),$$

is used, and the collocation points are chosen at the Fourier zeros $\left(\frac{1}{2}\cos(i\pi/10), \cos(j\pi/10)\right)$, $i, j = 0(1)10$. Table 2 gives the coefficients of approximation. As before the missing entries are less than 5×10^{-8} in magnitude.

The numerical results for the previous two examples were obtained by solving the resulting algebraic system of equations for the series coefficients by the standard elimination method.

Acknowledgments

The author would like to thank the referees for their valuable comments and suggestions, which have improved the original manuscript to its present form.

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